NEW UNIVARIATE SYNTHETIC DOUBLE SAMPLING $\bar{X}$ AND MULTIVARIATE SYNTHETIC DOUBLE SAMPLING T^2 CONTROL CHARTS

LEE HOW CHINH

UNIVERSITI SAINS MALAYSIA

2017

NEW UNIVARIATE SYNTHETIC DOUBLE SAMPLING $\bar{X}$ AND MULTIVARIATE SYNTHETIC DOUBLE SAMPLING T^2 CONTROL CHARTS

by

LEE HOW CHINH

Thesis submitted in fulfillment of the requirements

for the degree of

Doctor of Philosophy

February 2017

ACKNOWLEDGEMENT

First and foremost, I would like to express my sincere gratitude to my supervisor, Prof. Michael Khoo Boon Chong, for his continuous support and guidance in my Ph.D. study. I have benefited enormously from his moral support and insightful feedback, especially with his vast experience and immense knowledge in the field of Statistical Quality Control. Without his patience and motivation, this thesis would not have been completed. His constant encouragement and enthusiasm has helped me to overcome many obstacles during the research and writing of this thesis. I am so lucky to have such a great and friendly supervisor.

Besides my supervisor, I would also like to thank the Dean, Prof. Hailiza Kamarulhaili and all the lecturers and staff of the School of Mathematical Sciences, Universiti Sains Malaysia, for their kind advice and support during my Ph.D. study. The services, facilities and conducive environment provided are important factors that have enabled me to complete my Ph.D. study smoothly.

I am grateful to have the support of my family members, especially my mother, Lim Bee Cheng. Her support and blessings at all times gave me the energy and calmness to face the difficulties in life. Moreover, I am really fortunate to my wife, Tan Swee Lan, for her understanding and care, which has allowed me to continue my study without any worries. Without my daughter, Lee Zixuan, my family would not be complete. Her encouragement becomes a strong push when things went wrong.

Finally, thanks to all my friends and colleagues, who have either assisted me directly or indirectly in my Ph.D. study. Their help, suggestions and encouragements are crucial for the completion of this thesis.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS AND NOTATIONS

ANOS	Average number of observations to signal
ANOS(0)	In-control average number of observations to signal
$\mathrm{ANOS}(\delta)$	Out-of-control average number of observations to signal for univariate charts
$\mathrm{ANOS}(d)$	Out-of-control average number of observations to signal for multivariate charts
ARL	Average run length
ARL(0)	In-control average run length
$\mathrm{ARL}(\delta)$	Out-of-control average run length for univariate charts
$\mathrm{ARL}(d)$	Out-of-control average run length for multivariate charts
ARMA	Autoregressive moving average
ASS	Average sample size
$\mathrm{ASS}(0)$	In-control average sample size
cdf	Cumulative distribution function
CL	Center line
CRL	Conforming run length
CUSUM	Cumulative Sum
DS	Double sampling
EWMA	Exponentially weighted moving average
GR	Group runs
LCL	Lower control limit
MCUSUM	Multivariate cumulative sum
MEWMA	Multivariate exponentially weighted moving average
pdf	Probability density function

pmf	Probability mass function
SPC	Statistical Process Control
SSGR	Side sensitive group runs
SSGRDS	Side sensitive group runs double sampling
SSMGR	Side sensitive modified group runs
UCL	Upper control limit
VSI	Variable sampling interval
VSS	Variable sample size

Notations

n	Sample size
n_1	Size of the first sample
n_2	Size of the second sample
X_{1i}	Observation in the first sample
X_{2i}	Observation in the second sample
$\bar{X}_{1,i}$	Sample mean of the first sample, for sampling stage i
$\bar{X}_{2,i}$	Sample mean of the second sample, for sampling stage i
$\bar{\bar{X}}_i$	Grand average of the pooled sample mean, for sampling stage i
P_a	Probability that the process is in-control
P_{a1}	Probability that the process is in-control by the first sample
P_{a2}	Probability that the process is in-control by the second sample
$P_{a,1}$	Probability that the process is conforming by the first sample
$P_{a,2}$	Probability that the process is conforming by the second sample
$P_{\text{SDS}}(\delta)$	Probability of a non-conforming sampling stage on the DS sub-chart

$\mathbf{X}_{i,j}$	jth multivariate observation in sample i
$\bar{\mathbf{X}}_{k,1}$	Sample mean vector of the first sample in sampling stage k
$\bar{\mathbf{X}}_{k,2}$	Sample mean vector of the second sample in sampling stage k
$\bar{\bar{\mathbf{X}}}_{k}$	Pooled sample mean vector
$T_{k,1}^2$	T^2 statistic computed from the first sample
T_{k}^2	T^2 statistic computed from the combined samples
$\Phi(\cdot)$	Cumulative distribution function of the standard normal random variable
$\phi(\cdot)$	Probability density function of the standard normal random variable
$\Gamma(\cdot)$	Gamma function
$F_{\chi_p^2}(\cdot)$	Cumulative distribution function of a central chi-square random variable
$F_{\chi_p^2(\lambda)}(\cdot)$	Cumulative distribution function of a non-central chi-square random variable with non-centrality parameter λ
β	Probability of a Type-II error
α	Probability of a Type-I error
μ	Population mean
σ	Population standard deviation
δ	Size of the mean shift
δ_{opt}	Size of the mean shift, where a quick detection is important
$\bar{X}$	Sample mean
S	Sample standard deviation
$\mathbf{q}$	Initial probability vector
$\mathbf{I}$	Identity matrix

$\boldsymbol{\mu}$	Population mean vector
$\boldsymbol{\mu}_0$	In-control population mean vector
$\boldsymbol{\Sigma}$	Population covariance matrix
$\boldsymbol{\Sigma}_0$	In-control population covariance matrix
d	Mahalanobis distance
d_{opt}	Mahalanobis distance, where a quick detection is important
$\otimes$	Kronecker product
$\odot$	Element wise multiplication

Abbreviations used for the Shewhart chart

$\text{UCL}_{\bar{X}}$	Upper control limit of the $\bar{X}$ chart
$\text{CL}_{\bar{X}}$	Center line of the $\bar{X}$ chart
$\text{LCL}_{\bar{X}}$	Lower control limit of the $\bar{X}$ chart
UCL_R	Upper control limit of the R chart
CL_R	Center line of the R chart
LCL_R	Lower control limit of the R chart
UCL_S	Upper control limit of the S chart
CL_S	Center line of the S chart
LCL_S	Lower control limit of the S chart
$\text{ARL}_{\bar{X}}$	Average run length for the $\bar{X}$ chart
$\text{ANOS}_{\bar{X}}$	Average number of observations to signal for the $\bar{X}$ chart

Abbreviations for the EWMA chart

UCL_{EWMA}	Upper control limit
LCL_{EWMA}	Lower control limit
ARL_{EWMA}	Average run length
$\text{ANOS}_{\text{EWMA}}$	Average number of observations to signal

Notation for the CRL chart

L	Lower control limit

Abbreviations and notations for the synthetic chart

L'	Lower limit of the CRL sub-chart
$\text{UCL}_{\bar{X}/S}$	Upper control limit of the $\bar{X}$ sub-chart
$\text{LCL}_{\bar{X}/S}$	Lower control limit of the $\bar{X}$ sub-chart
$\text{ARL}_{\text{synthetic}}$	Average run length
$\text{ANOS}_{\text{synthetic}}$	Average number of observations to signal

Abbreviations and notations for the DS chart

L_1	Upper warning limit of the chart for the first sample
$-L_1$	Lower warning limit of the chart for the first sample
L	Upper control limit of the chart for the first sample
$-L$	Lower control limit of the chart for the first sample
L_2	Upper control limit of the chart for the second sample
$-L_2$	Lower control limit of the chart for the second sample
ARL_{DS}	Average run length

$\mathrm{ASS}_{\mathrm{DS}}$	Average sample size
$\mathrm{ANOS}_{\mathrm{DS}}$	Average number of observations to signal

Abbreviations and notations for the synthetic DS $\bar{X}$ chart

L_1	Upper warning limit of the DS sub-chart for the first sample
$-L_1$	Lower warning limit of the DS sub-chart for the first sample
L	Upper control limit of the DS sub-chart for the first sample
$-L$	Lower control limit of the DS sub-chart for the first sample
L_2	Upper control limit of the DS sub-chart for the second sample
$-L_2$	Lower control limit of the DS sub-chart for the second sample
L_3	Lower control limit of the CRL sub-chart
$\mathrm{ARL}_{\mathrm{SDS}}$	Average run length
$\mathrm{ASS}_{\mathrm{SDS}}$	Average number of observations to signal
$\mathrm{ANOS}_{\mathrm{SDS}}$	Average sample size

Abbreviations for the Hotelling's T^2 chart

$\mathrm{UCL}_{\text{Hotelling's } T^2}$	Upper control limit
$\mathrm{ARL}_{\text{Hotelling's } T^2}$	Average run length
$\mathrm{ANOS}_{\text{Hotelling's } T^2}$	Average number of observations to signal

Abbreviations for the MEWMA chart

$\mathrm{UCL}_{\mathrm{MEWMA}}$	Upper control limit
$\mathrm{ARL}_{\mathrm{MEWMA}}$	Average run length
$\mathrm{ANOS}_{\mathrm{MEWMA}}$	Average number of observations to signal

Abbreviations and notations for the multivariate synthetic T^2 chart

L_{CRL}	Lower control limit of the CRL sub-chart
UCL_{T^2}	Upper control limit of the T^2 sub-chart
ARL_{MS}	Average run length
ANOS_{MS}	Average number of observations to signal

Abbreviations and notations for the multivariate DS T^2 chart

w_1	Warning limit
h_1	Control limit
ARL_{MDS}	Average run length
ASS_{MDS}	Average sample size
ANOS_{MDS}	Average number of observations to signal

Abbreviations and notations for the multivariate synthetic DS T^2 chart

w_1	Upper warning limit of the DS T^2 sub-chart for the first sample
h_1	Upper control limit of the DS T^2 sub-chart for the first sample
h	Upper control limit of the DS T^2 sub-chart for the second sample
L_{CRL}	Lower control limit of the CRL sub-chart
ARL_{MSDS}	Average run length
ASS_{MSDS}	Average sample size
$\text{ANOS}_{\text{MSDS}}$	Average number of observations to signal

CARTA-CARTA KAWALAN $\bar{X}$ PENSAMPELAN GANDA DUA SINTETIK UNIVARIAT DAN T^2 PENSAMPELAN GANDA DUA SINTETIK MULTIVARIAT BARU

ABSTRAK

Tesis ini mencadangkan carta kawalan $\bar{X}$ pensampelan ganda dua sintetik univariat untuk memantau min proses univariat dan carta kawalan T^2 pensampelan ganda dua sintetik multivariat untuk memantau vektor min proses multivariat. Carta univariat yang dicadangkan menggabungikan carta panjang larian patuhan (CRL) dan carta $\bar{X}$ pensampelan ganda dua (DS) manakala carta mutivariat yang dicadangkan menggabungkan carta CRL dan carta DS T^2. Kedua-dua carta univariat dan multivariat yang dicadangkan adalah lebih unggul daripada carta-carta asas masing-masing, iaitu carta-carta $\bar{X}$ sintetik dan DS $\bar{X}$ bagi kes univariat, dan carta-carta T^2 sintetik dan DS T^2 bagi kes multivariat, daripada segi kriteria prestasi panjang larian purata (ARL) dan bilangan cerapan purata untuk memberi isyarat (ANOS). ANOS adalah kriteria prestasi yang lebih bermakna berbanding dengan ARL untuk carta-carta dengan saiz sampel boleh berubah, seperti carta-carta yang dicadangkan dan carta DS. Ini kerana ANOS mengukur bilangan cerapan sebenar yang diperlukan sebelum isyarat luar kawalan dikeluarkan oleh carta. Tafsiran prestasi carta berdasarkan ARL adalah rumit apabila keputusan tentang prestasi carta dibuat berdasarkan sampel pertama sahaja atau kedua-dua sampel pertama dan kedua. Carta DS $\bar{X}$ sintetik yang dicadangkan telah ditunjukkan mengatasi carta asasnya yang sepadan, iaitu carta-carta $\bar{X}$ sintetik dan DS $\bar{X}$. Dengan membandingkan carta DS $\bar{X}$ sintetik dengan carta purata bergerak berpemberat eksponen (EWMA), carta yang terdahulu memberi prestasi yang lebih baik daripada carta yang terkemudian,

bagi julat saiz anjakan min yang lebih besar, berbanding dengan carta-carta $\bar{X}$ sintetik dan DS $\bar{X}$. Carta DS $\bar{X}$ sintetik mempunyai prestasi yang lebih baik daripada carta EWMA untuk mengesan anjakan min yang sederhana dan besar, manakala carta yang terkemudian mengatasi carta yang terdahulu untuk mengesan anjakan yang kecil. Carta DS T^2 sintetik multivariat juga adalah lebih unggul daripada carta purata bergerak berpemberat eksponen multivariat (MEWMA) untuk mengesan anjakan sederhana dan besar. Walau bagaimanapun, carta MEWMA adalah lebih peka daripada carta DS T^2 sintetik multivariat untuk mengesan anjakan kecil.

NEW UNIVARIATE SYNTHETIC DOUBLE SAMPLING $\bar{X}$ AND MULTIVARIATE SYNTHETIC DOUBLE SAMPLING T^2 CONTROL CHARTS

ABSTRACT

This thesis proposes a univariate synthetic double sampling $\bar{X}$ control chart to monitor the mean of a univariate process and a multivariate synthetic double sampling T^2 control chart for monitoring the mean vector of a multivariate process. The proposed univariate chart integrates the conforming run length (CRL) chart and the double sampling (DS) $\bar{X}$ chart, while the proposed multivariate chart combines the CRL chart and the DS T^2 chart. Both the proposed univariate and multivariate charts are superior to their basic counterparts, namely the synthetic $\bar{X}$ and DS $\bar{X}$ charts for the univariate case, and the synthetic T^2 and DS T^2 charts for the multivariate case, in terms of the average run length (ARL) and average number of observations to signal (ANOS) performance criteria. ANOS is a more meaningful performance criterion compared to the ARL for the variable sample size charts, such as the proposed charts and the DS chart. This is because ANOS measures the actual number of observations required before an out-of-control signal is issued by the chart. Interpretation of a chart's performance based on ARL is complicated when a decision about a chart's performance is made based on either the first sample only or both the first and second samples. The proposed synthetic DS $\bar{X}$ chart is shown to outperform its basic counterparts, i.e. the synthetic $\bar{X}$ and DS $\bar{X}$ charts. By comparing the synthetic DS $\bar{X}$ chart with the exponentially weighted moving average (EWMA) chart, the former provides a better performance than the latter, for a larger range of mean shift sizes, compared with the synthetic $\bar{X}$ and DS $\bar{X}$ charts.

The synthetic DS $\bar{X}$ chart has a better performance than the EWMA chart for detecting moderate and large mean shifts, while the latter overtakes the former for detecting small shifts. The multivariate synthetic DS T^2 chart is also superior to the multivariate exponentially weighted moving average (MEWMA) chart for detecting moderate and large shifts. However, the MEWMA chart is more sensitive than the multivariate synthetic DS T^2 chart for detecting small shifts.

CHAPTER 1

INTRODUCTION

1.1 Statistical Process Control

Statistical Process Control (SPC) is one of the quantitative tools in Total Quality Management. The seven SPC tools comprise the Pareto diagram, cause and effect diagram, check sheet, process flow diagram, scatter diagram, histograms and control charts (Besterfield, 2009). These seven tools are often called "The magnificent seven" (Montgomery, 2012).

SPC deals with the quality of conformance. It involves examining the results of a process or service with a standard, and taking corrective actions if discrepancy exists between the two (Mitra, 2008). The improvements in a process, such as uniformity of output, minimizing rework, fewer defective products, increased profitability, lower average cost, reducing errors, improving output quality, reducing scrapped cost, minimizing machine downtime, increasing job satisfaction and improving competitive position, can be obtained by applying SPC (Smith, 1991). Therefore, a proper deployment of SPC helps in establishing a continuous improvement in the quality and productivity of an organization (Montgomery, 2012).

To minimize the number of defective products and reduce variability, corrective actions must be taken on a real time basis when the output deviates from the standard. Therefore, the information of a product or a process needs to be collected while it is functional. This attempt is called online SPC (Mitra, 2008)

1.2 Control Charts

Among "the magnificent seven", a control chart is one of the primary techniques in SPC and it is the simplest type of an on-line SPC procedure. The idea of a control chart was proposed by Walter A. Shewhart of the Bell Telephone Labortories in 1924, to control the variables of a product (Mitra, 2008). Shewhart realized that the enemy of quality is variation and that variation in a process can be divided into common causes and assignable causes (Sower, 2009). A control chart is a graphical display of a certain descriptive statistic for a specific quantitative measurement in a manufacturing process. The main function of a control chart is to identify the type of variation that is present in a process.

The basic feature of a control chart is a time sequence plot with decision lines. The decision lines are crucial in determining whether a process is statistically in control. A control chart is constructed based on the sampling distribution of the inspected descriptive statistics (Ryan, 2000).

Basically, there are two types of control charts, i.e. control charts for variables data and control charts for attributes data. Variables controls charts are control charts for central tendency and variability, and attributes control charts are control charts for quality characteristics, like the number of nonconformities or the fraction of nonconforming items (Montgomery, 2012).

1.2.1 Univariate Control Charts

The common univariate variables control chart used for monitoring the process mean is the $\bar{X}$ chart. A control chart for the range, i.e. the R chart or a control chart for the standard deviation, i.e. the S chart, is commonly used to monitor process variability. The $\bar{X}$, R and S charts are mainly used for detecting large shifts quickly (Montgomery, 2012). On the other hand, the univariate variables control charts which are more effective for detecting small shifts in the process mean or variance are the cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) type charts (Mitra, 2008). The discussion from here onwards deal with the synthetic and double sampling charts, as these charts are the focus of this thesis.

The $\bar{X}$ chart and the conforming run length (CRL) chart were integrated by Wu and Spedding (2000) to construct the synthetic $\bar{X}$ chart. The synthetic $\bar{X}$ chart significantly outperforms the $\bar{X}$ chart, for all levels of mean shifts and it provides a better detection ability than the EWMA $\bar{X}$ chart, for a standardized mean shift of greater than 0.8 standard deviation. Since its inception, many studies on synthetic charts were made. The robustness of the synthetic $\bar{X}$ chart to non-normality was studied by Calzada and Scariano (2001). Davis and Woodall (2002) presented a Markov chain model of the synthetic $\bar{X}$ chart and used it to evaluate the zero-state and steady-state ARL performances. Scariano and Calzada (2003) presented a synthetic chart for exponential data. Sim (2003) worked on combining the $\bar{X}$ and CRL charts by assuming that the quality characteristic follows a gamma distribution. The synthetic charts for process dispersion were also studied extensively. The R and CRL charts were integrated by Chen and Huang (2005), while Huang and Chen (2005) combined the S and CRL charts to produce synthetic charts for process dispersion.

Chen and Huang (2006) integrated the Max and CRL charts to produce the synthetic Max chart for a joint monitoring of the process mean and standard deviation. Costa and Rahim (2006) suggested a synthetic chart to jointly monitor the process mean and variance, based on the non-central chi-square statistics, where the chart can detect a shift more effectively than the joint $\bar{X}$ and R chart. Khoo et al. (2008), and Castagliola and Khoo (2009) introduced synthetic charts for skewed populations, based on the weighted variance and scaled weighted variance methods, respectively. Aparisi and de Luna (2009) developed a synthetic chart which only detects shifts that are considered important and does not detect shifts in a region of admissible shifts. Scariano and Calzada (2009) proposed generalized synthetic charts by considering the EWMA and CUSUM charts. Zhang et al. (2011) studied the performance of the synthetic $\bar{X}$ chart with estimated process parameters using the Markov chain approach. They suggested guidelines on the recommended number of Phase-I samples and the sample sizes besides providing new optimal charting constants, particularly for the sample size used in practice.

Acceptance sampling is an optimal option between the extremes of 100% inspection and no inspection. Usually, the information obtained from the sample is used to make a decision as to whether to reject or accept the entire lot of products (Montgomery, 2012). There are several types of sampling plans, namely the single-sampling plan, double-sampling plan and multiple sampling plan (Mitra, 2008). The concept of the double sampling (DS) $\bar{X}$ chart, which is based on the double sampling plan was first studied by Croasdale (1974). Daudin (1992) proposed a DS control chart, where the information from both first and second samples is used to make a decision about the process at the combined samples stage. The DS $\bar{X}$ chart gives a better statistical efficiency than the $\bar{X}$ chart, in terms of the average run length (ARL), without increasing the sample size. The DS $\bar{X}$ chart can be used to reduce the average sample size without penalizing the statistical efficiency.

Numerous works on the DS $\bar{X}$ charts have been suggested in the literature. Irianto and Shinozaki (1998) proposed an optimal DS $\bar{X}$ chart that minimizes the out-of-control ARL of the chart to detect shifts in the process mean. The DS S chart for the process standard deviation in agile manufacturing was developed by He and Grigoryan (2002). Genetic algorithm was applied by He et al. (2002) to design the DS $\bar{X}$ chart and to study its performance. Carot et al. (2002) combined the double sampling method with the variable sampling interval feature, where the new chart is found to be more sensitive for detecting small and moderate shifts in the process mean compared with the CUSUM and EWMA charts. He and Grigoryan (2003) improved on the DS S chart so that it does not require the normality assumption of the underlying distribution. This improvement is very useful when the sample size is small. The DS S^2 chart was proposed by Khoo (2004) for monitoring the process variance. He and Grigoryan (2006) developed a joint statistical design of the DS $\bar{X}$ and S charts. Costa and Claro (2008) considered a DS $\bar{X}$ chart to monitor a process with observations from a first-order autoregressive moving average (ARMA(1,1)) model. Torng and Lee (2009a) presented a modified statistical design of the DS $\bar{X}$ chart based on the multi-objective programming method and genetic algorithm. The performance of a DS $\bar{X}$ chart under non-normality was studied by Torng and Lee (2009b). They found that it outperforms the $\bar{X}$ chart for detecting small process mean shifts but it has a comparable performance with the variable parameter $\bar{X}$ chart.

1.2.2 Multivariate Control Charts

The control charts mentioned in Section 1.2.1 deal with the monitoring of only one variable. Nevertheless, two or more variables need to be controlled simultaneously in real process monitoring situations in manufacturing. Controlling the different variables separately may not produce a product in which all the variables are acceptable and monitoring correlated data independently can be very misleading (Mitra, 2008).

Multivariate control charts are used to monitor several related variables of interest simultaneously. Multivariate quality control was initiated by Hotelling (1947), who applied the multivariate procedure to bombsight data during World War II. Multivariate quality control is particularly important in today's manufacturing environment. Automatic inspection procedures provide opportunities for manufacturers to measure several parameters on each product manufactured and to maintain the manufacturing databases in routine, even when the process or the quality data involve hundreds of variables and the total size of these databases is measured in millions of individual records. It is ineffective to monitor or analyze these data with univariate SPC procedures (Montgomery, 2012).

Generally, multivariate control charts can be categorized into three main categories, namely, Hotelling's T^2 charts, multivariate EWMA (MEWMA) charts and multivariate cumulative sum (MCUSUM) charts. The Hotelling's T^2 chart is only based on the current sample, whereas the MEWMA and MCUSUM charts accumulate information from the past and present observations. Therefore, the MEWMA and MCUSUM charts are more sensitive for detecting small and moderate shifts in the mean vector of a multivariate process compared with the Hotelling's T^2 chart. The Hotelling's T^2 chart is a direct extension of the univariate $\bar{X}$ chart, and it is the most common multivariate SPC procedure for monitoring the mean vector of a multivariate process (Montgomery, 2012).

In recent years, research works on the Hotelling's T^2 type charts include extensive studies on the synthetic T^2 and DS T^2 charts. Costa and Machado (2007) suggested a synthetic control chart with two stage sampling for monitoring bivariate processes. Ghute and Shrike (2008a) presented a synthetic T^2 chart to monitor the mean vector of a process that follows a multivariate normal distribution. The synthetic T^2 chart has significantly better detection ability than the Hotelling's T^2 chart and the T^2 chart with supplementary runs rules. Ghute and Shrike (2008b) also introduced a multivariate synthetic $|\boldsymbol{S}|$ chart which is based on the generalized sample variance, $|\boldsymbol{S}|$ statistic. The multivariate synthetic $|\boldsymbol{S}|$ chart is used to monitor the covariance matrix of a multivariate normally distributed process. They showed that this chart performs better than the standard $|\boldsymbol{S}|$ chart. Machado et al. (2009a) suggested the VMAX and synthetic VMAX charts for monitoring the covariance matrix, which are based on two sample variances. These two charts are more effective than the generalized sample variance $|\boldsymbol{S}|$ chart. Khoo et al. (2009) presented a multivariate synthetic chart for monitoring the mean vector of skewed populations using weighted standard deviations. This chart has a better performance than the existing multivariate charts for skewed populations. Machado et al. (2009b) proposed a single chart, called the synthetic MVMAX chart to monitor the mean vector and covariance matrix of bivariate processes. The points plotted on the MVMAX chart correspond to the maximum of the sample means and variances of the quality characteristics.

On the other hand, the multivariate double sampling $|\boldsymbol{S}|$ chart, for monitoring shifts in the covariance matrix was proposed by Grigoryan and He (2005). Their results showed that for certain out-of-control situations, the performance of this chart is among those of the best schemes considered. He and Grigoryan (2005) developed a statistical design optimization procedure for the multivariate double sampling chart for the process mean. Costa and Machado (2008) proposed a double sampling T^2 chart to control bivariate processes. This chart performs better than the standard bivariate T^2 chart and the adaptive bivariate T^2 chart with variable sample size (VSS) and variable sampling interval (VSI). Champ and Aparisi (2008) introduced two double sampling T^2 charts to monitor the mean vector of a multivariate process with p dimensions. These two charts differ on how the second sample is used to decide on the state of a process. These charts perform relatively well, in terms of the ARL criterion compared with other selected multivariate charts.

1.3 Objectives of the Study

Two main objectives in this thesis are

(i) to propose a univariate synthetic double sampling $\bar{X}$ chart, for the process mean, which is superior to the existing $\bar{X}$ type charts, in terms of the ARL and the average number of observations to signal (ANOS) criteria.

(ii) to propose a multivariate synthetic double sampling T^2 chart, for the process mean vector, which has better ARL and ANOS performances than the existing T^2 type charts.

1.4 Methodologies of the Study

This thesis provides a thorough discussion on the proposed univariate and multivariate control charts. For the univariate case, the proposed synthetic DS $\bar{X}$ chart, which comprises the DS $\bar{X}$ sub-chart and the CRL sub-chart, is an extension of the DS $\bar{X}$ chart of Daudin (1992) and the CRL chart of Bourke (1991). An optimization procedure is developed to obtain the optimal parameters of the synthetic DS $\bar{X}$ chart, for minimizing the out-of-control ARL and out-of-control ANOS. The optimization procedure is written in the Mathematica 7 program. The performance of the synthetic DS $\bar{X}$ control chart is evaluated and compared with its standard counterparts, such as the DS $\bar{X}$, synthetic $\bar{X}$ and EWMA $\bar{X}$ charts.

The multivariate case is similar to the univariate case, where the synthetic DS T^2 chart which consists of the DS T^2 sub-chart and the CRL sub-chart is constructed by merging the DS T^2 chart of Champ and Aparisi (2008) and the CRL chart of Bourke (1991). The optimal design of the synthetic DS T^2 chart is obtained by minimizing the out-of-control ARL and out-of-control ANOS. The optimal parameters of the synthetic DS T^2 chart are computed from an optimization program written using SAS 9.3. The performance of the synthetic DS T^2 chart is compared with its standard counterparts, like the Hotelling's T^2, DS T^2, synthetic T^2 and MEWMA charts.

1.5 Organization of the Thesis

Chapter 1 gives a brief discussion on SPC, as well as on univariate and multivariate control charts. It also explains the objectives and methodologies of the study. In Chapter 2, the univariate distributions and univariate control charts which will be used in the later chapters are discussed. The multivariate normal distribution and multivariate control charts, such as the Hotelling's T^2, MEWMA, synthetic T^2 and DS T^2 charts are discussed in Chapter 3, as they are required in the later chapters. Chapter 4 provides a detailed discussion on the proposed univariate synthetic DS $\bar{X}$ chart for the process mean. A performance evaluation of the synthetic DS $\bar{X}$ chart and an example of application are presented in this chapter. Chapter 5 presents the proposed multivariate synthetic DS T^2 chart for the process mean vector together with a performance evaluation and an illustrative example to show how the chart is implemented. Finally, Chapter 6 wraps up this thesis by providing a conclusion and some suggestions for further research.

CHAPTER 2

A REVIEW ON SELECTED UNIVARIATE DISTRIBUTIONS AND UNIVARIATE CONTROL CHARTS

2.1 Introduction

Several univariate distributions that are important in the construction of control charts and will be used in the later chapters are discussed in this chapter. These statistical distributions are the normal, geometric and chi-square distributions.

A control chart can be categorized based on the type of the quality characteristic of interest. A desired variable can be measured on a numerical scale or registered on a categorical scale. A control chart based on data measured on a numerical scale or on a categorical scale is called control chart for variables or control chart for attributes, respectively. One of the most important control charts for variables data is the Shewhart control chart. The Shewhart type control charts consist of the $\bar{X}$, *R* and *S* charts, which are widely used to monitor the mean and variability of a process. On the other hand, the commonly used attribute charts are the *p*, *np*, *c* and *u* charts. The conforming run length (CRL) chart is another attribute chart that is considered in this thesis. It monitors the number of conforming items in a process.

For decades, improvements on the Shewhart $\bar{X}$ control chart has been made in two main directions, i.e. to increase its speed for detecting small and moderate shifts in the process mean and to reduce the average sample size needed in maintaining the efficiency of the chart. The synthetic and double sampling control charts were suggested to address these issues.

2.2 Univariate Distributions

The constructions of most control charts are based on probability distributions. Some univariate distributions will be discussed in this section.

2.2.1 The Normal Distribution

The normal distribution is one of the most popular continuous distributions and it is widely used in control charts. The probability density function (pdf) of the normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \text{ for } -\infty < x < \infty, \tag{2.1}$$

where μ and σ are the mean and standard deviation of the distribution, respectively (Ryan, 2000). The cumulative distribution function (cdf) of the normal distribution can be obtained by finding the probability that the normal random variable X is less than or equal to a certain value, say a, hence

$$\Pr(X \le a) = F(a) = \int_{-\infty}^{a} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx. \tag{2.2}$$

Since the above definite integral cannot be evaluated in closed form, the change of variable $Z = \frac{X-\mu}{\sigma}$ can be employed to make the evaluation be independent of μ and σ. Therefore,

$$\Pr(X \le a) = \Pr\left(Z \le \frac{a-\mu}{\sigma}\right) \equiv \Phi\left(\frac{a-\mu}{\sigma}\right), \tag{2.3}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution (Montgomery, 2012). The μ of the normal distribution is a location parameter, where a change in the mean causes a shift in the location of the distribution. On the other hand, the σ of the normal distribution is a shape parameter, where a change in the standard deviation will change the spread of the distribution about the mean.

The pdf of a normal distribution is bell-shaped, unimodal and symmetry about μ. According to Montgomery (2012), a simple interpretation of the standard deviation, σ, of a normal distribution is as follows: 68.26% of the population values fall within the one standard deviation interval from the mean, the two standard deviations interval from the mean contains 95.46% of the population values, and 99.73% of the population values fall within the three standard deviations interval. These percentages are significant for the construction of the Shewhart control chart (Mitra, 2008).

2.2.2 The Geometric Distribution

The geometric distribution is a discrete distribution, where its probability represents the first occurrence of a success at the x^{th} independent trial. The success probability for each independent trial is denoted as p. The probability mass function (pmf) of a geometric random variable, X is

$$g(x;p) = pq^{x-1}, \quad \text{for } x = 1, 2, \ldots \quad \text{where } q = 1 - p. \tag{2.4}$$

The mean and variance of a geometrically distributed random variable X are $\frac{1}{p}$ and $\frac{1-p}{p^2}$, respectively (Hogg et al., 2005).

The assumption of using a geometric distribution is that a system stays in the same state for all the trials. With this property, the geometric distribution is considered as a lack of memory discrete distribution (Montgomery and Runger, 2007). The geometric distribution is a fundamental distribution for the CRL chart.

2.2.3 The Chi-Square Distribution

The chi-square distribution is a special case of the gamma distribution. The gamma distribution is a family of distributions defined by the following pdf:

$$f(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}, \qquad x > 0, \tag{2.5}$$

where $\alpha > 0$ and $\beta > 0$ are the parameters of the distribution and $\Gamma(\cdot)$ refers to the gamma function,

$$\Gamma(\alpha) = \int_0^{\infty} e^{-y} y^{\alpha-1} dy. \tag{2.6}$$

Note that $\Gamma(n) = (n-1)!$ (Hogg et al., 2005).

The chi-square distribution is obtained when $\alpha = \frac{r}{2}$ and $\beta = 2$ in Equation (2.5), which gives

$$f(x) = \frac{1}{2^{r/2}\Gamma\left(\frac{r}{2}\right)} x^{\frac{r}{2}-1} e^{-\frac{x}{2}}, \qquad x > 0, \tag{2.7}$$

where *r* is the degrees of freedom of the chi-square distribution. If a random variable *X* is distributed with a chi-square distribution, then it can be denoted as $X \sim \chi^2(r)$. The mean and variance of a chi-square random variable are *r* and 2*r*, respectively (Hogg et al., 2005).

The following theorem establishes a relationship between the standard normal and chi-square variables. If $Z \sim N(0,1)$, then $Z^2 \sim \chi^2(1)$. Hence, if the random variables $X_1, X_2, ..., X_n$ are from the $N(\mu, \sigma^2)$ distribution, then $\sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2} \sim \chi^2(n)$ (Bain and Engelhardt 1992). In this sense, the chi-square distribution is a crucial distribution for deriving a control chart's statistics, where the underlying distribution satisfies the normality assumption.

2.3 Univariate Control Charts

In this session, five univariate control charts, which will be employed in a later chapter will be discussed. These five charts are the Shewhart, EWMA, CRL, synthetic and double sampling $\bar{X}$ charts.

2.3.1 The Shewhart Control Chart

As the name of this chart implies, Walter Shewhart of Bell Labs initiated the general idea of this chart in 1924 (Ryan, 2000). This chart can be considered as the formal beginning of Statistical Quality Control charts.

The general model of the Shewhart chart is based on the probability distribution of the sample statistic that measures the quality characteristic. Suppose that w is the sample statistic, the mean of w is μ_w and the standard deviation of w is σ_w. Then the center line (CL), upper control limit (UCL) and lower control limit (LCL) of the Shewhart chart are

$$\text{UCL} = \mu_w + L\sigma_w \tag{2.8a}$$

$$\text{CL} = \mu_w \tag{2.8b}$$

$$\text{LCL} = \mu_w - L\sigma_w, \tag{2.8c}$$

where L is the width of the control limits, in terms of multiples of standard deviations from the CL (Montgomery, 2012).

In the monitoring of a quality characteristic that is variable, it is important that both the process mean and the process variability are monitored. Monitoring of the process mean is usually conducted using the $\bar{X}$ chart, while the process variability is monitored by either a chart for the range (*R* chart) or a chart for the standard deviation (*S* chart).

To construct the $\bar{X}$ chart, suppose that the quality characteristic, X is normally distributed with mean μ and standard deviation σ, i.e. $X \sim N\left(\mu, \sigma^2\right)$, where both μ and σ are known. For a sample of size *n*, its sample mean is computed as

$$\bar{X} = \frac{X_1 + X_2 + ... + X_n}{n} . \tag{2.9}$$

According to the sampling distribution of X_i, for $i = 1, 2, ..., n$, $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$. Suppose that $\bar{X}_1, \bar{X}_2, ..., \bar{X}_m$ are the means of *m* preliminary samples, then by referring to Equations (2.8a), (2.8b) and (2.8c), the UCL, CL and LCL of the three sigma $\bar{X}$ chart are

$$\mathrm{UCL}_{\bar{X}} = \mu + 3\frac{\sigma}{\sqrt{n}} \tag{2.10a}$$

$$\mathrm{CL}_{\bar{X}} = \mu \tag{2.10b}$$

$$\mathrm{LCL}_{\bar{X}} = \mu - 3\frac{\sigma}{\sqrt{n}} . \tag{2.10c}$$

To develop the $\bar{X}$ and *R* charts, the following unbiased estimator of σ is needed:

$$\hat{\sigma} = \frac{\bar{R}}{d_2} , \tag{2.11}$$

where $\bar{R}$ is the average sample range. Note that the sample range is computed as $R = X_{\max} - X_{\min}$, where $X_{\max}$ and $X_{\min}$ represent the largest and smallest observations in the sample. By substituting σ for $\hat{\sigma}$ and μ for $\bar{\bar{X}}$ in Equations (2.10a) - (2.10c), the limits of the $\bar{X}$ chart are

$$\mathrm{UCL}_{\bar{X}} = \bar{\bar{X}} + 3\frac{\bar{R}}{d_2\sqrt{n}} = \bar{\bar{X}} + A_2\bar{R} \tag{2.12a}$$

$$\mathrm{CL}_{\bar{X}} = \bar{\bar{X}} \tag{2.12b}$$

$$\mathrm{UCL}_{\bar{X}} = \bar{\bar{X}} - 3\frac{\bar{R}}{d_2\sqrt{n}} = \bar{\bar{X}} - A_2\bar{R}, \tag{2.12c}$$

where $A_2 = \dfrac{3}{d_2\sqrt{n}}$ and its value for an arbitrary sample size n is given in most quality control text books (Montgomery, 2012). Note that $\bar{\bar{X}}$ is the grand sample average of $\bar{X}_1, \bar{X}_2, ..., \bar{X}_m$.

The average run length (ARL) is defined as the average number of samples taken before a shift is detected and it can be expressed as $\mathrm{ARL} = \dfrac{1}{\Pr\left(\text{one point plots out-of-control}\right)}$. If the process is in-control, then the in-control ARL of the $\bar{X}$ chart is $\mathrm{ARL}_{\bar{X}}\left(0\right) = \dfrac{1}{\alpha}$, where α is the probability of a Type-I error. The out-of-control ARL of the $\bar{X}$ chart is given by (Montgomery, 2012)

$$\mathrm{ARL}_{\bar{X}}\left(\delta\right) = \frac{1}{1-\beta}, \tag{2.13}$$

where β is the probability of a Type-II error, having a process shift, $\delta\left(>0\right)$, in the process mean. Here, $\beta = \Phi\left(3 - \dfrac{\delta\sqrt{n}}{\sigma}\right) - \Phi\left(-3 - \dfrac{\delta\sqrt{n}}{\sigma}\right)$. Hence, the average number of observations to signal (ANOS) for the $\bar{X}$ chart is

$$\mathrm{ANOS}_{\bar{X}}\left(\delta\right) = n \times \mathrm{ARL}_{\bar{X}}\left(\delta\right), \tag{2.14}$$

where $\text{ARL}_{\bar{X}}(\delta)$ is given in Equation (2.13) (Montgomery, 2012).

The limits of the *R* chart are given as follows (Montgomery, 2012):

$$\text{UCL}_R = D_4\bar{R} \tag{2.15a}$$

$$\text{CL}_R = \bar{R} \tag{2.15b}$$

$$\text{LCL}_R = D_3\bar{R}, \tag{2.15c}$$

where $D_4 = 1 + \frac{3d_3}{d_2}$ and $D_3 = \max\left(0,\ 1 - \frac{3d_3}{d_2}\right)$. The values of D_3 and D_4 can also be found in most quality control text books (Montgomery, 2012).

Although the $\bar{X}$ and *R* charts are popular and widely used in industries, their counterparts, $\bar{X}$ and *S* charts are preferable when the sample size is moderately large or the sample size varies (Montgomery, 2012). Furthermore, the range, *R* is calculated from only two observations in a sample, thus the information from the range does not involve all the elements in the sample. This loss of information can be serious when the sample size is large (Ryan, 2000).

If the variance of the population, σ^2 is unknown, an unbiased estimator of σ^2 is the sample variance,

$$S^2 = \frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}{n-1}. \tag{2.16}$$

However, the sample standard deviation, *S* is not an unbiased estimator of the population standard deviation, σ. Montgomery (2012) pointed out that if the distribution of the population is normally distributed, then the sample standard deviation, *S* is an unbiased estimator of $c_4\sigma$, where c_4 is a constant that depends on the sample size, *n*. The standard deviation of *S* is $\sigma\sqrt{1-c_4^2}$. Thus, the UCL, CL and LCL of the *S* chart are

$$\mathrm{UCL}_S = c_4\sigma + 3\sigma\sqrt{1-c_4^2} = B_6\sigma \tag{2.17a}$$

$$\mathrm{CL}_S = c_4\sigma \tag{2.17b}$$

$$\mathrm{LCL}_S = c_4\sigma - 3\sigma\sqrt{1-c_4^2} = B_5\sigma, \tag{2.17c}$$

where $B_5 = c_4 - 3\sqrt{1-c_4^2}$ and $B_6 = c_4 + 3\sqrt{1-c_4^2}$.

If σ is unknown, then it needs to be estimated from past data. If *m* preliminary samples, each of size *n* are available, then the average sample standard deviation is computed as

$$\bar{S} = \frac{1}{m}\sum_{i=1}^{m} S_i, \tag{2.18}$$

where S_i is the standard deviation of sample *i*, for $i = 1, 2, ..., m$. As $\frac{\bar{S}}{c_4}$ is an unbiased estimator of σ, the limits of the *S* chart are

$$\mathrm{UCL}_S = \bar{S} + 3\frac{\bar{S}}{c_4}\sqrt{1-c_4^2} = B_4\bar{S} \tag{2.19a}$$

$$\mathrm{CL}_S = \bar{S} \tag{2.19b}$$

$$\mathrm{LCL}_S = \bar{S} - 3\frac{\bar{S}}{c_4}\sqrt{1-c_4^2} = B_3\bar{S}, \tag{2.19c}$$

where $B_3 = 1 - \frac{3}{c_4}\sqrt{1-c_4^2}$ and $B_4 = 1 + \frac{3}{c_4}\sqrt{1-c_4^2}$.

Using $\frac{\bar{S}}{c_4}$ as an unbiased estimator of σ and $\bar{\bar{X}}$ as an unbiased estimator of μ, the limits of the $\bar{X}$ chart in Equations (2.10a) - (2.10c) can be redefined as

$$\mathrm{UCL}_{\bar{X}} = \bar{\bar{X}} + 3\frac{\bar{S}}{c_4\sqrt{n}} = \bar{\bar{X}} + A_3\bar{S} \tag{2.20a}$$

$$\mathrm{CL}_{\bar{X}} = \bar{\bar{X}} \tag{2.20b}$$

$$\text{LCL}_{\bar{X}} = \bar{\bar{X}} - 3\frac{\bar{S}}{c_4\sqrt{n}} = \bar{\bar{X}} - A_3\bar{S}, \tag{2.20c}$$

where $A_3 = \dfrac{3}{c_4\sqrt{n}}$.

Although the S chart is preferable to the R chart in certain situations as mentioned earlier and when the computations are computerized, the S chart also has some weaknesses as the distribution of S is not symmetry when the distribution of the variable being monitored is normal (Ryan, 2000).

2.3.2 The Exponentially Weighted Moving Average (EWMA) Control Chart

The EWMA chart was introduced by Roberts (1959) and was further studied by Crowder (1989), and Lucas and Saccuci (1990). It is a good alternative to the Shewhart control chart when the detection of small shifts is a concern. The EWMA chart is recommended for individual measurements and it is a robust or distribution free procedure, hence, it is insensitive to the normality assumption (Montgomery, 2012).

The EWMA chart's statistic is defined as

$$Z_i = \lambda\bar{X}_i + \left(1-\lambda\right)Z_{i-1}, \tag{2.21}$$

where $0 < \lambda \leq 1$ is a weight assigned to the sample average at time i, $\bar{X}_i$. The starting value of the EWMA statistic is the process target or the center line, $Z_0 = \mu_0$. Normally, the iterative calculations start with $Z_0 = \bar{\bar{X}}$, the grand average of the introductory data (Ryan, 2000).

Suppose that the sample means $\bar{X}_1, \bar{X}_2, \ldots$, are independent of each other and the population variance is σ^2 . Then according to Montgomery (2012), the variance of Z_i is

$$\sigma_{Z_i}^2 = \sigma^2\left(\frac{\lambda}{2-\lambda}\right)\left(1-(1-\lambda)^{2i}\right). \quad (2.22)$$

Hence, according to Equations (2.8a)-(2.8c) the limits of the EWMA chart are

$$\text{UCL} = \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2i}\right]} \quad (2.23a)$$

$$\text{CL} = \mu_0 \quad (2.23b)$$

$$\text{LCL} = \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2i}\right]}, \quad (2.23c)$$

where L is a constant controlling the width of the control limits.

The control limits will approach a constant when i becomes larger. This is because $\lim_{i\to\infty}\left[1-(1-\lambda)^{2i}\right]=1$. In short, the UCL/LCL of the EWMA chart will approach the following constant values after numerous iterations:

$$\text{UCL}_{\text{EWMA}} = \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} \quad (2.24a)$$

$$\text{LCL}_{\text{EWMA}} = \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} \quad (2.24b)$$

The values of λ and L are selected based on the desired in-control and out-of-control ARL performances (Montgomery, 2012).

From Zhang et al. (2009), the ARL of the EWMA chart can be expressed as

$$\text{ARL}_{\text{EWMA}} = \mathbf{q}^{\text{T}}(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{1}, \quad (2.25)$$

where $\mathbf{Q}$ is the $p\times p$ matrix of transient probabilities, $\mathbf{q}^{\text{T}} = (q_1, q_2, ..., q_p)$ is the $1\times p$ vector of initial probabilities associated with the p transient states, $\mathbf{I}$ is the identity matrix and $\mathbf{1}^{\text{T}} = (1,1,...,1)$. To evaluate ARL_{EWMA}, The interval between UCL and LCL is divided into $p = 2m+1$ subintervals, each of width 2η, where

$\eta = \dfrac{\text{UCL} - \text{LCL}}{2p}$. Let H_j, for $j = -m, ..., -1, 0, +1, ..., +m$, be the midpoint of the *j*th subinterval. Thus, the generic element $Q_{i,j}$ which is the transient probability in the matrix **Q** is given by (Zhang et al., 2009)

$$Q_{i,j} = \Phi\left[\left(\frac{H_j + \eta - (1-\lambda)H_i}{\lambda} - a\right)\frac{\sqrt{n}}{b}\right] - \Phi\left[\left(\frac{H_j - \eta - (1-\lambda)H_i}{\lambda} - a\right)\frac{\sqrt{n}}{b}\right]. \quad (2.26)$$

Note that the underlying process observations, X_i, for $i = 1, 2, ...$, follow a $N\left(\mu_0 + a\sigma,\ b^2\sigma^2\right)$ distribution. When $a \neq 0$ and $b \neq 1$, the process mean and standard deviation have shifted. The generic element q_j, for $j = 1, 2, ..., p$ of vector **q** is given by

$$q_j = \begin{cases} 1, & \text{if } H_j - \eta < Z_0 < H_j + \eta \\ 0, & \text{otherwise} \end{cases}. \quad (2.27)$$

When the number of subintervals, *p*, is sufficiently large, it allows the ARL_{EWMA} to be accurately evaluated. Then the ANOS of the EWMA chart is

$$\text{ANOS}_{\text{EWMA}} = n \times \text{ARL}_{\text{EWMA}}. \quad (2.28)$$

2.3.3 The Conforming Run Length Control Chart

The conforming run length (CRL) chart was suggested by Bourke (1991) to monitor attribute data. The CRL is defined as the number of inspected units between two successive nonconforming units, where the count includes the ending nonconforming unit. For example, in Figure 2.1 the CRL is 5.

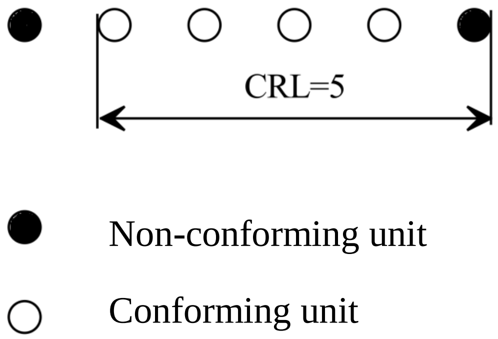

Figure 2.1 Conforming run length

The CRL behaves according to a geometric distribution, where the random variable CRL will change when the fraction nonconforming in the process, p changes. The CRL control chart is used to detect increases in the fraction nonconforming, p. Thus, a single lower control limit, L is sufficient to fulfill this objective, where

$$L = \frac{\ln(1-\alpha)}{\ln(1-p_0)}. \tag{2.29}$$

Note that p_0 is the in-control fraction nonconforming while α is the size of the Type-I error given by (Wu et al., 2001)

$$\alpha = 1-(1-p_0)^L. \tag{2.30}$$

If a CRL value is less than or equal to L, the process fraction nonconforming, p has increased.

2.3.4 The Synthetic $\bar{X}$ Control Chart

The synthetic control chart was proposed by Wu and Spedding (2000) to monitor shifts in the process mean. This chart comprises a Shewhart $\bar{X}$ sub-chart and a CRL sub-chart. An out-of-control signal is not generated based on the $\bar{X}$ sub-chart alone. An $\bar{X}$ sample plotted beyond the control limits of the $\bar{X}$ sub-chart will only denote a nonconforming sample. The information from the $\bar{X}$ sub-chart will be transferred to the CRL sub-chart as either conforming or nonconforming samples. An out-of-control signal will only be generated once the CRL value is smaller than or equal to L', which is the lower limit of the CRL sub-chart.

The upper and lower control limits of the $\bar{X}$ sub-chart are (Wu and Spedding, 2000),

$$\mathrm{UCL}_{\bar{X}/S} = \mu_0 + k\sigma_{\bar{X}} \tag{2.31a}$$

$$\mathrm{LCL}_{\bar{X}/S} = \mu_0 - k\sigma_{\bar{X}}, \tag{2.31b}$$

where μ_0 is the in-control process mean and $\sigma_{\bar{X}}$ is the standard deviation of the sample mean. k is the control limit coefficient of the $\bar{X}$ sub-chart and it is usually smaller than the control limit coefficient of the corresponding Shewhart $\bar{X}$ chart. The ARL of a synthetic chart, $\mathrm{ARL}_{\mathrm{synthetic}}(\delta)$ denotes the average number of the $\bar{X}$ samples needed to detect a process shift, δ, in the mean and it is given by (Wu and Spedding, 2000)

$$\mathrm{ARL}_{\mathrm{synthetic}}(\delta) = \frac{1}{P} \times \frac{1}{1-(1-P)^{L'}}, \tag{2.32}$$

where

$$P = 1 - \Phi\left(k - \delta\sqrt{n}\right) + \Phi\left(-k - \delta\sqrt{n}\right). \tag{2.33}$$

If the process is in-control, i.e. $\delta = 0$, the in-control ARL is

$$\mathrm{ARL}_{\mathrm{synthetic}}(0) = \frac{1}{2\Phi(-k)} \times \frac{1}{1-\left(1-2\Phi(-k)\right)^{L'}}. \tag{2.34}$$

Then the ANOS of the synthetic chart can be evaluated as

$$\mathrm{ANOS}_{\mathrm{synthetic}} = n \times \mathrm{ARL}_{\mathrm{synthetic}}. \tag{2.35}$$

Equation (2.34) is used to determine the two parameters, k and L' needed to construct the $\bar{X}$ sub-chart and CRL sub-chart, respectively. The main idea for this